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Olivier Brüls*, Vincent Acary†, and Alberto Cardona°

*University of Liège, Department of Aerospace and Mechanical Engineering (LTAS), Liège, Belgium

†INRIA Rhône-Alpes, Centre de recherche Grenoble, St Ismier, France

° Universidad Nacional del Litoral - Conicet, CIMEC, Santa Fe, Argentina

Summary. In this work, we present a formalism for the numerical time integration of nonsmooth dynamical systems composed of rigid and flexible bodies, kinematic joints and frictionless contact conditions. The proposed algorithm guarantees the exact satisfaction of the bilateral and unilateral constraints both at position and velocity levels, extending a previous work on the development of the generalized-α scheme for computational contact mechanics. Following the idea of Gear, Gupta and Leimkuhler, the equation of motion is reformulated so that the bilateral and unilateral constraints appear both at position and velocity levels which amounts to solving two complementarity conditions at both position and velocity levels at each iteration using a monolithic semi–smooth Newton procedure.

Introduction and Motivations

This paper studies numerical algorithms for the simulation of mechanical systems including rigid and flexible bodies, kinematic joints and frictionless contact conditions. The condition of impenetrability of the bodies in contact is expressed as a unilateral constraint, with the consequence that impacts and/or instantaneous changes in the velocities may arise in the dynamic response.

The algorithm proposed here for systems with rigid and/or flexible bodies is inspired from the Moreau–Jean time-stepping strategy [8]. A fundamental property of the Moreau–Jean scheme is that the unilateral constraints are imposed at velocity level. As recognized by many authors, this choice leads to interesting consistency and stability properties of a simulation algorithm for dynamic contact analysis. In the Moreau–Jean scheme, the unilateral constraints at position level are only used to support the decision to activate the constraint but they are not exactly satisfied. The consequence is that some penetration can be observed in the numerical solution, which may not be physically acceptable. In order to prevent such penetration problems, this paper presents an algorithm which enforces the constraint not only at velocity level, so as to inherits good consistency and stability properties, but also at position level. For that purpose, the Gear-Gupta-Leimkuhler (GGL) approach [7], which was initially developed for the stabilization of index-3 differential-algebraic equations, is generalized to systems with unilateral constraints following a similar idea as in [1].

Another property of the Moreau–Jean algorithm is that the complete system is integrated in time using a method which is only first–order accurate. This appears as a disadvantage in applications where the nonsmooth phenomena are localized in some mechanical parts of the system, while other parts exhibit smooth motion, for example, the dynamic response of wind turbines systems with the smooth motions and vibrations of the blades and the tower, and nonsmooth dynamic phenomena in the gearbox and in the transmission line. In this case, the description of vibration of phenomena in flexible bodies with a first-order method would require quite small time steps and would thus be inefficient. In order to improve the accuracy in the smooth part of motion, Chen et al [6] observed that some terms in the equations of motion of a nonsmooth dynamic system, such as the elastic forces, are smooth and can be integrated in the time domain using a higher-order scheme, e.g., the generalized-α method [5, 4]. All impulsive terms are still treated using a fully implicit integration scheme to ensure consistency. This means that different integration formulae are used for the different contributions to the equation of motion.

Time–continuous setting

After spatial semi-discretization, the equations of motion of a flexible multibody system including bilateral and unilateral constraints can be expressed in the following form:

$$\begin{cases} \dot q = v^+, \qquad M(q)\,\mathrm{d}v - g_q^\top \mathrm{d}i = f(t,q,v)\mathrm{d}t & \text{(1a)}\\ g^{\overline{\mathcal{U}}}(q) = 0 & \text{(1b)}\\ 0 \leq g^{\mathcal{U}}(q) \perp \mathrm{d}i \geq 0 & \text{(1c)}\\ \text{if } g^j(q(t_i)) \leq 0, \text{ then } 0 \leq g_q^j v^+(t_i) + e g_q^j v^-(t_i) \perp \mathrm{d}i^j \geq 0, \text{for all } j \in \mathcal{U}, & \text{(1d)} \end{cases}$$

where q is the vector of coordinates, e.g., absolute nodal coordinates, $v = v^+$ is the vector of velocities, f collects the external, damping and internal forces, $M(q)$ is the mass matrix, $\mathrm{d}v$ is the differential measure associated with the velocity v assumed to be of bounded variation, $\mathrm{d}t$ is the standard Lebesgue measure, g is the combined set of bilateral and unilateral constraints, and g_q is the corresponding matrix of constraint gradients, $\mathrm{d}i$ is the impulse measure of the contact reaction and the bilateral force. Finally, the index sets $\mathcal{U}$ denotes the set of indices of the unilateral constraints, $\overline{\mathcal{U}}$ is its complementarity set, i.e., the set of bilateral constraints, $\mathcal{C} = \mathcal{U} \cup \overline{\mathcal{U}}$ is the full set of constraints. For more details, we refer to [1, 2, 6].

Following a similar approach as in [6], a splitting of the variables and of the equations of motion is proposed to isolate impulsive terms, which are integrated with only first order accuracy, from smooth contributions, which shall be integrated using a higher order method. For a given time step $(t_n, t_{n+1}]$, the position $\tilde{q}(t)$, velocity $\tilde{v}(t)$ and Lagrange multiplier $\tilde{\lambda}(t)$ are solutions of the following initial value problem coupled with the variables $q(t)$ and $v(t)$

$$\begin{cases} \dot{\tilde{q}} = \tilde{v}, \qquad M(q)\,\dot{\tilde{v}} - g_q^\top \tilde{\lambda} = f(t,q,v) & \text{(2a)} \\ g_q^{\overline{\mathcal{U}}}\tilde{v} = 0, \quad \tilde{\lambda}^{\mathcal{U}} = 0. & \text{(2b)} \end{cases}$$

Then, the variable $\mathrm{d}w$ is defined by the equation $\mathrm{d}w = \mathrm{d}v - \tilde{v}\mathrm{d}t$. The motivation for this splitting is to express dv as the sum of a smooth contribution which is represented by the variable $\tilde{v}$ and of some impulsive contributions which are all collected by the variable $\mathrm{d}w$. In terms of these variables, the equation of motion becomes

$$\begin{cases} \dot{q} = v, & \text{(3a)} \\ M(q)\,\dot{\tilde{v}} - g_q^\top \tilde{\lambda} = f(t,q,v), \qquad M(q)\mathrm{d}w - g_q^\top(\mathrm{d}i - \tilde{\lambda}\mathrm{d}t) = 0, \qquad \mathrm{d}v = \mathrm{d}w + \dot{\tilde{v}}\mathrm{d}t & \text{(3b)} \\ g_q^{\overline{\mathcal{U}}}\tilde{v} = 0, \quad \tilde{\lambda}^{\mathcal{U}} = 0, \quad g_q^{\overline{\mathcal{U}}} v = 0 & \text{(3c)} \\ \text{if } g^j(q) \le 0, \text{ then } 0 \le g_q^j v + e g_q^j v^- \perp \mathrm{d}i^j \ge 0, \text{for all } j \in \mathcal{U}. & \text{(3d)} \end{cases}$$

We propose to include the constraints at velocity and position levels in a unique set of equations following a similar method as proposed in [7]. The equilibrium equation is then obtained from (3) after replacing (3a) by

$$\begin{cases} M(q)\dot{q} - g_q^\top \mu = M(q)v & \text{(4a)} \\ g^{\overline{\mathcal{U}}}(q) = 0, \quad 0 \le g^{\mathcal{U}}(q) \perp \mathrm{d}i^{\mathcal{U}} \ge 0. & \text{(4b)} \end{cases}$$

By construction, any solution of (3) also satisfies (4) with $\mu = 0$.

Time–discretization

Starting from (3) and (4) , a time integration scheme is formulated to advance the solution at each time step as follows

$$\begin{cases} M(q_{n+1})\dot{\tilde{v}}_{n+1} - f(q_{n+1}, v_{n+1}, t_{n+1}) - g_{q,n+1}^{\overline{\mathcal{U}},\top}\tilde{\lambda}^{\overline{\mathcal{U}}} = 0 & \text{(5a)} \\ M(q_{n+1})W_{n+1} - g_{q,n+1}^\top \Lambda_{n+1} = 0 & \text{(5b)} \\ M(q_{n+1})U_{n+1} - g_{q,n+1}^\top \nu_{n+1} = 0 & \text{(5c)} \\ g_{q,n+1}^{\overline{\mathcal{U}}}\tilde{v}_{n+1} = 0, \qquad g_{q,n+1}^{\overline{\mathcal{U}}} v_{n+1} = 0, \qquad g^{\overline{\mathcal{U}}}(q_{n+1}) = 0, \qquad 0 \le g^{\mathcal{U}}(q_{n+1}) \perp \nu_{n+1}^{\mathcal{U}} \ge 0 & \text{(5d)} \\ \text{if } g^j(q_{n+1}) \le 0, \text{ then } 0 \le g_{q,n+1}^j v_{n+1} + e g_{q,n}^j v_n^- \perp \Lambda_{n+1}^j \ge 0, \text{for all } j \in \mathcal{U}, & \text{(5e)} \end{cases}$$

with the adapted integration formulae of the generalized-α scheme

$$\begin{cases} \tilde{q}_{n+1} = q_n + h v_n + h^2(1/2 - \beta)a_n + h^2\beta a_{n+1}, \qquad q_{n+1} = \tilde{q}_{n+1} + U_{n+1} & \text{(6a)} \\ \tilde{v}_{n+1} = v_n + h(1-\gamma)a_n + h\gamma a_{n+1}, \qquad v_{n+1} = \tilde{v}_{n+1} + W_{n+1} & \text{(6b)} \\ (1-\alpha_m)a_{n+1} + \alpha_m a_n = (1-\alpha_f)\dot{\tilde{v}}_{n+1} + \alpha_f \dot{\tilde{v}}_n, & \text{(6c)} \end{cases}$$

where the numerical coefficients γ, β, α_m and α_f can be selected from a desired value of the spectral radius at the infinity [4]. After the proposed time discretization, the equations of motion involve two complementarity conditions and the problem can be solved efficiently at each time step using a monolithic Newton semi-smooth method [3]. The complementarity conditions are imposed using an active set method, i.e., the activation criteria are evaluated at each Newton iteration in a fully implicit way. The numerical examples demonstrate the fast and robust convergence of the Newton semi-smooth procedure.

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